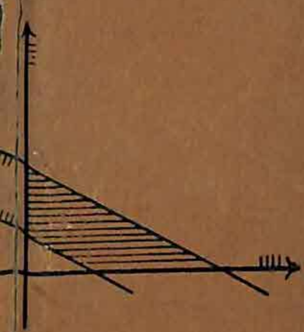

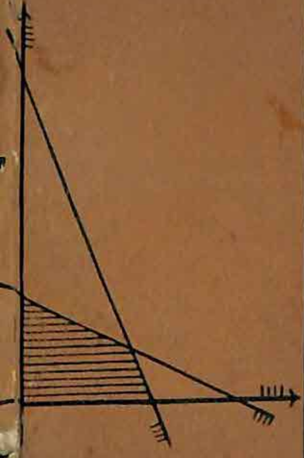

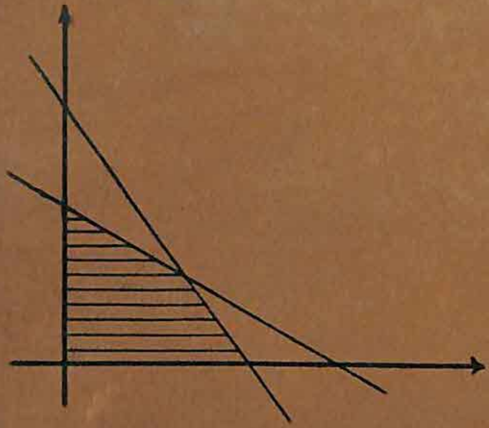

S. K. GUPTA

LINEAR PROGRAMMING FOR SCHOOLS

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LINEAR PROGRAMMING
FOR
SCHOOLS

By

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FOREWORD

Linear Programming has found a place in all improved school programmes all over the world. In India also, it is hoped, it will find a place in our newly developing curricula. I suggested to Dr. Gupta to write a book to facilitate this process and I am glad thet he readily accepted my suggestion.

The reasons for the popularity of linear programming in school curricula are three-fold (*i*) It is an interesting mathematical discipline in its own right (*ii*) it has important applications in industry (*iii*) its mathematical level is such that it can be taught at the school stage, specially because its theory can be developed without using calculus. Its importance can be judged from the simple fact that the annual bill paid by American industry alone for linear programming comes to several million dollars.

Most courses in linear programming at school level do not give any proofs and give only a heuristic treatment of the subject. This book presents a course which is significantly different from others in the sense that it gives a proof of the fundamental theorem of linear programming for the special case of two variables at a level that the school students can understand. Most practical problems involve a large number of variables and such large-scale problems can be done with the help of electronic computers only. In fact practical linear programming systems have been solved involving a million variables and thirty thousand equations.

Our industry has also started making use of linear programming on a large scale and it is but proper that our school students should become familiar with this discipline at as early a stage as possible.

I am sure the present book will be found equally useful by teachers and students.

J. N. KAPUR

IIT/Kanpur
May 1, 1968

PREFACE

During the past few years, mathematics teachers all over the world have been trying very hard to present *New Mathematics* to school students. The question that naturally crops up is: What is the necessity of so doing? The answer is: There exists a very vast literature of mathematics today which can be introduced at school level and which is far more realistic, useful, intersting and even challenging than the mathematics which we particularly in India, are teaching to our school students. Obviously then, it is interesting to dig out the mathematics which falls under this category. This book is a partial attempt in this direction; in as much as, *Linear Programming* is proposed to be taught at school stage, and the way it is to be taught is the urpose, pin fact, of this book.

It is obvious to anybody that when we are thinking of introducing some *New Mathematics* at school stage, we have as well to think about the mathematics that has to be discarded from the present curricula. Arithmetic, as it is presented today, is the greatest victim, because, apart from serving the purposes of drill in the four fundamental operations (which can be achieved otherwise in a much better way), most of the arithmetic that we are teaching serves no other purpose. Linear programming, I believe, is a good substitute for some portions of arithmetic which can safely be discarded. At some of our places, we have recently introduced *probability theory* as a compulsory course at higher secondary stage. It is suggested that linear programming should be offered as an alternative optional course, if drastic changes in the curricula are not sought.

The pre-requisites for the introduction of this course are : a little proficiency in algebra and a few results from the coordinate geometry of a straight line in a plane (these are reviewed briefly in Chapter I of the book). It is earnestly hoped that linear programming will soon find an appropriate place in our school curricula.

Chapter I is devoted to help the students recall the result from coordinate geometry which will be used later on. Chapter II is, in fact, the heart of the book, in which we prove the *fundamental theorem* of linear programming (for two variables only). Finally, chapter III is devoted to applications of the fundamental theorem to some simple problems which when formulated mathematically reduce to two variable linear programming problems.

I take this opportunity to thank Professor J.N. Kapur, Head of the department of mathematics, Indian Institute of Technology, Kanpur, for encouraging me to write this book.

Any suggestions for the improvement of the book are most welcome.

Department of Mathematics,
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Kanpur

S.K. GUPTA

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I

Linear Equations and Inequations

1. Introduction

The problem that we propose to tackle in this pamphlet may be stated formally as follows :

Maximise (or minimise)

$$ax+by$$

subject to

$$\begin{aligned} a_ix+b_iy \{\leqslant \text{ or } \geqslant \text{ or} =\}\ c_i\ ,\ i&=1,2,\ldots\ldots\ldots\ldots,m \\ x &\geqslant 0, \qquad \ldots(\text{A}) \\ \text{and} \qquad y &\geqslant 0, \end{aligned}$$

where m is some fixed natural number ; a, b, a_i, b_i, c_i for $i=1, 2,\ldots\ldots\ldots\ldots, m$ are fixed real numbers ; and x and y are real variables restricted by the conditions (A).

As we shall show in this chapter, conditions of the type contained in (A) specify a set of points enclosed by a polygon (closed or open) in a plane ; and our problem as stated above is then to find out that point (or points) from this set which maximise (or minimise) the expression $ax+by$.

The expression, $ax+by$, which is sought to be maximised (or minimised) is called the *objective function* ; and the conditions contained in (A) are called *restrictions*. The problem, as we have posed above, is called a linear programming problem because the objective function as also all the restrictions are linear.

The present chapter is introductory in nature; inasmuch as we shall review the graphical procedure only of presenting the set of points in a plane determined by the restrictions of the type (A). The actual procedure of tackling the linear programming problem will, however, be given in chapter 2.

Finally, some applications of linear programming will be considered in chapter 3.

2. Two Linear Equations in Two Unknowns. Graphical Solution Procedure.

Consider the problem of solving the following system of two linear equations in two unknowns :

$$a_1x+b_1y=c_1 \quad \dots \quad (i)$$

$$a_2x+b_2y=c_2 \quad \dots \quad (ii)$$

where a_1, b_1, c_1, a_2, b_2, c_2 are fixed real numbers and x and y are real variables*.

From co-ordinate geometry, we know that each one of these equations represents a straight line in a plane.

We also know that any point on the line representing $a_1x+b_1y=c_1$ is a *solution* and that the points on the line are the *only solutions* of this equation. A similar statement can be made about the equation $a_2x+b_2y=c_2$ also. Thus, we may state the following result : The *solution set* of any linear equation in two unknowns is the set of all points in a plane which lie on the line representing that equation.

---

*All constants and variables that appear now onwards are to be taken as reals unless something explicitly is mentioned otherwise.

We recall now that solving two equations simultaneously is equivalent to finding out the solution (or solutions) which satisfy both the equations simultaneously. Thus, if we know the solution sets of the individual equations, then the problem of simultaneous solution of the two equations is equivalent to finding out the common elements of the two solution sets, *i.e.*, the *intersection* of the two solution sets. In case both the equations are linear equations, then the individual solution sets are represented by straight lines in a plane; and, therefore, the simultaneous solution of two linear equations in two unknowns is represented by the point (or points) of intersection of the two lines.

Next, we observe that a system of two linear equations in two unknowns has

(*i*) a unique solution

(*ii*) an infinite number of solutions

(*iii*) no solution at all

according as the lines representing the equations of the system are

(*i*) neither parallel nor coincident

(*ii*) coincident

(*iii*) parallel but not coincident.

Let us recall that the lines represented by the equations (*i*) and (*ii*) are parallel if

$$a_1 b_2 - a_2 b_1 = 0$$

and are coincident if

$$a_1 b_2 - a_2 b_1 = 0$$

and $a_1 c_2 - a_2 c_1 = 0,$

$$b_1 c_2 - b_2 c_1 = 0.$$

Thus the system of equations

$$a_1x+b_1y=c_1$$
$$a_2x+b_2y=c_2$$

has

(*i*) a unique solution
(*ii*) an infinite number of solutions
(*iii*) no solution at all

according as

(*i*) $a_1b_2-a_2b_1 \neq 0$ (This condition ensures that the lines cannot be coincident)
(*ii*) $a_1b_2-a_2b_1=a_1c_2-a_2c_1=b_1c_2-b_2c_1=0$
(*iii*) $a_1b_2-a_2b_1=0$,
$a_1c_2-a_2c_1 \neq b_1c_2-b_2c_1$.

As examples, consider the following systems of equations :

$$\left.\begin{aligned} 2x+3y&=4 \\ x+y&=2 \end{aligned}\right\} \quad \dots \quad (1)$$

$$\left.\begin{aligned} 5x+6y&=30 \\ 15x+18y&=90 \end{aligned}\right\} \quad \dots \quad (2)$$

$$\left.\begin{aligned} 4x+9y&=16 \\ 16x+36y&=40 \end{aligned}\right\} \quad \dots \quad (3)$$

The graphs of these equations are given in figs. 1.1, 1.2 and 1.3.

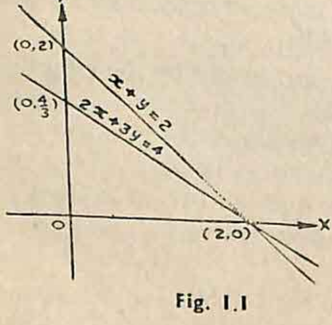

Fig. 1.1

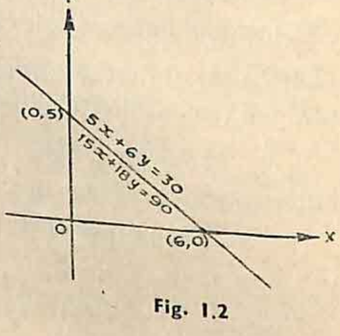

Fig. 1.2

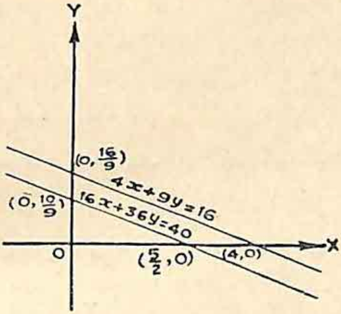

Fig. 1·3.

Thus system (1) has the unique solution at the point (2, 0) *i.e.*, $x=2, y=0$; system (2) has an infinite number of solutions (any point on the line representing the equation $5x+6y=30$); and, system (3) has no solution at all.

2·1 More than two linear equations in two unknowns

Let us now consider the problem of solving the following three linear equations in two unknowns :

$$a_1x+b_1y=c_1 \qquad \ldots \quad (i)$$

$$a_2x+b_2y=c_2 \qquad \ldots \quad (ii)$$

$$a_3x+b_3y=c_3. \qquad \ldots \quad (iii)$$

Leaving aside the obviously uninteresting case when two or more of the equations are identical, we give in figs. 1·4, 1·5, 1·6 and 1.7 the possible behaviour of the lines representing these three equations.

Thus, we see that in case I (fig. 1.4) only a solution exists, and that, too, uniquely. Moreover, we may also observe that an infinite number of solutions can exist only if all the three equations are identical. In all other cases, no solution exists at all.

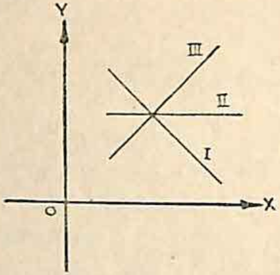

Fig. 1·4.

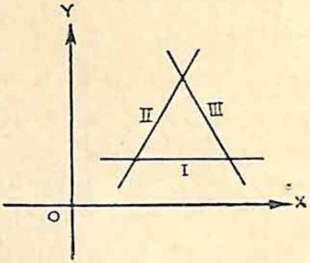

Fig. 1·5.

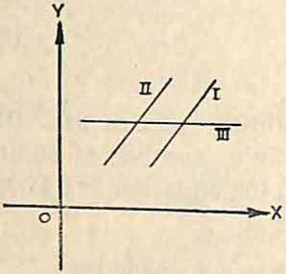

Fig. 1·6.

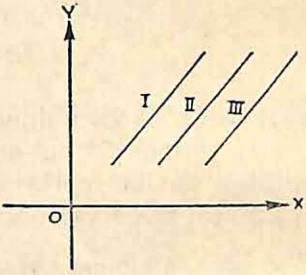

Fig. 1·7.

We may now verify the truth of the following statements :

Given *any* number of linear equations in two unknowns ;

(1) A *unique solution* exists if and only if the lines representing these equations are concurrent ;

(2) An infinite number of solutions exists if and only if all the equations are identical ; and

(3) No solution exists in all other cases.

3. Linear Inequations

As we can talk of the solution set of an equation, we can as well talk of the solution set of an inequation of any one of the forms

$$2x+5y \geqslant 6 \quad \text{or} \quad 2x+5y \leqslant 6$$

or

$$2x+5y > 6 \quad \text{or} \quad 2x+5y < 6$$

Now we observe that this straight line divides the whole plane into three naturally disjoint sets, namely

(*i*) the set of points *on the line*

$$ax+by=c\ ;$$

(*ii*) the set of points shaded by horizontal line ; and

(*iii*) the set of points shaded by vertical lines.

Now from co-ordinate geometry, we know that the set 'B' of all points (x, y) for which $ax+by-c$ is positive lie on *one side* of the line and the set, C, of points (x, y) for which $ax+by-c$ is negative lie on the *other side* of the line.

Thus, if we can determine the position of a single point in relation to the line $ax+by=c$, we can at once single out the solution set of $ax+by < c$. The most convenient point to this effect is the origin (0, 0).

If we substitute $x=0$, $y=0$ in $ax+by < c$, we get $0 < c$, which may be a true or a false statement, depending upon whether (0, 0) is in the solution set or not. Thus, we have :

The solution set of the inequality $ax+by < c$ is the side of the line $ax+by=c$

(*i*) which contains (0, 0) if $0 < c$, or

(*ii*) which does not contain (0, 0) if $0 > c$.

We may now make similar statements for the inequalities

$$ax+by \geqslant c,\ ax+by \leqslant c,\ ax+by > c.$$

To fix up the ideas, we illustrate the procedure by means of a specific simple example.

Suppose that we are interested in finding out the solution set of the inequality

$$2x+5y < 6.$$

in the sense that the solution set of the inequality, say,

$$2x+5y > 6$$

is the set 'A' of all ordered pairs (x, y) such that when the values of x and y as specified by *any* element of A are substituted in $2x+5y > 6$, we get a true statement. Thus, $(4, 0)$ and $(0, 2)$ belong to the set A, because

$2 \cdot 4+5 \cdot 0 > 6$ is a true statement and $0 \cdot 2+5 \cdot 2 > 6$ is a true statement. Similarly, we can find *many more* elements of the solution set A of the inequality $2x+5y > 6$.

3·1. Graphical Solution Procedure

To find out the solution set of the inequality

$$ax+by \leqslant c$$

we will now describe a graphical solution procedure.

We know that the linear equation

$$ax+by=c$$

represents a straight line in a plane. Let us draw this line as shown in fig. 1·8.

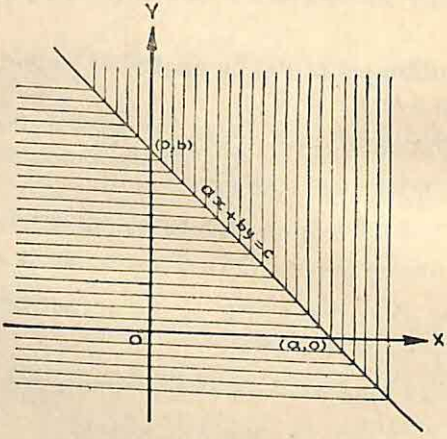

Fig. 1·8.

Let us draw the line

$$2x+5y=6.$$

Now, putting $x=0$, $y=0$ in the given inequality, we find that

$$2\cdot0+5\cdot0 < 6 \quad \text{i.e.} \quad 0 < 6$$

which is a true statement. Therefore (0, 0) belongs to the solution set of $2x+5y < 6$. Hence the portion of the line containing (0, 0), shaded in fig. 1.9 (excluding the points *on* the line), is the solution set.

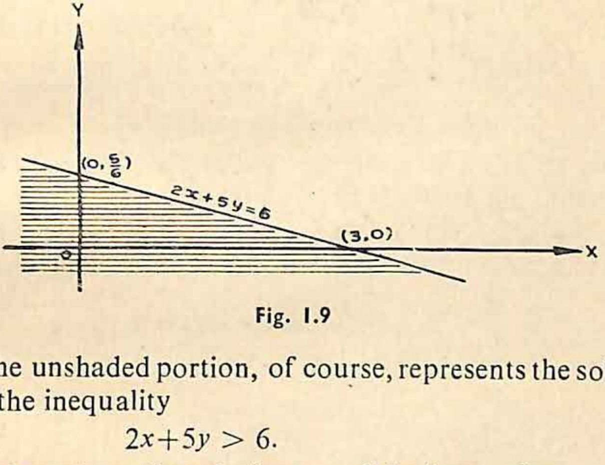

Fig. 1.9

The unshaded portion, of course, represents the solution set of the inequality

$$2x+5y > 6.$$

Now what about the solution set of the inequality

$$2x+5y \leqslant 6\ ?$$

This, obviously, is the union of the solution set of the inequality $2x+5y < 6$ and the solution set of the equation $2x+5y=6$, *i.e.* the set of all shaded points and the points on the line $2x+5y=6$, in the figure.

4. System of Linear Inequations

Now suppose that we are interested in solving simultaneously a system of linear inequations in two unknowns.

As in the case of a system of linear equations, the solution set of a system of linear inequations is given by the intersection of the solution sets of individual inequations of the system. Thus, to fond the solution set of a system of linear inequations, we first find the solution sets of the individual inequations of the system, and then take the intersection of these solution sets thus obtained. The procedure is explained below by means of a specific example.

Suppose that we are interested in finding out the solution set of the following system of linear inequations :

$$\begin{aligned} 5x+12y &\geqslant 20 \\ 3x+8y &\leqslant 40 \\ x &\geqslant 0 \\ y &\geqslant 0. \end{aligned}$$

The solution sets of the individual inequalities are exhibited in fig. 1·10.

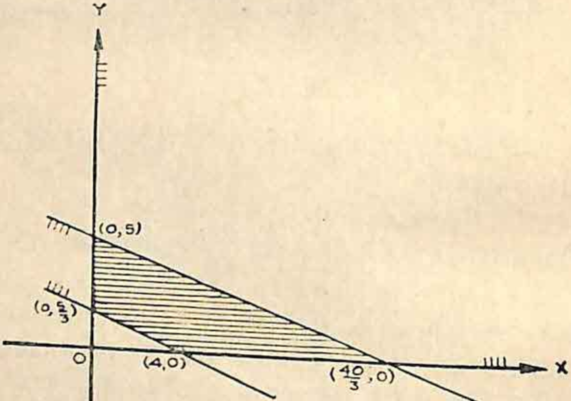

Fig. 1.10.

Now, $x \geqslant 0$ and $y \geqslant 0$ indicates that we have to consider the points in the first quadrant only. The solution set of

Now, $\{tx_1+(1-t)x_2,\ ty_1+(1-t)y_2\}$ will belong to the solution set if and only if

$$a[tx_1+(1-t)x_2]+b[ty_1+(1-t)y_2] \leqslant c$$

or
$$t(ax_1+by_1)+(1-t)\ (ax_2+by_2) \leqslant c \qquad \dots\ (iii)$$

For any value of t, such that $0 \leqslant t \leqslant 1$, we see that $t \geqslant 0$ and $1-t \geqslant 0$. Therefore, multiplying (*i*) by t and (*ii*) by $(1-t)$, we have

$$tax_1+tby_1 \leqslant tc \qquad \dots\ (iv)$$
$$(1-t)ax_2+(1-t)by_2 \leqslant (1-t)\ c \qquad \dots\ (v)$$

Adding (*iv*) and (*v*), we get (*iii*).

This proves the theorem.

Note : This result is also valid for any system of linear inequaiities, as may be very easily proved as above.

Theorem 2. *The co-ordinates of any point on the line segment joining two points* (x_1, y_1) *and* $(x_1,\ y_2)$ *are of the from*

$$\{tx_1+(1-t)x_2,\ ty_1+(1-t)y_2\}, \text{ where } 0 \leqslant t \leqslant 1.$$

Proof. Let P and Q be the points (x_2, y_1) and (x_2, y_2) respectively. Draw perpendiculars PL and QM on x-axis and QC on PL.

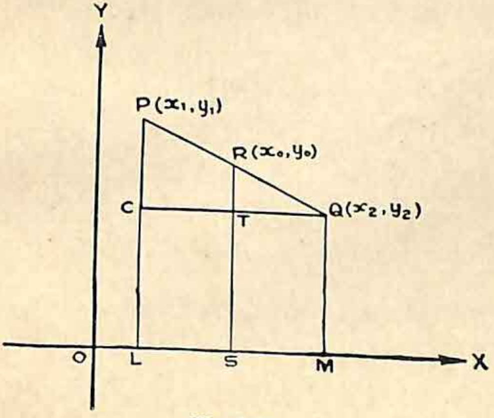

Fig. 1·11.

$5x+12y \geqslant 20$ is the side of the line $5x+12y=40$ which does not contain the origin ; and the solution set of $3x+8y \leqslant 40$ is the side of the line $3x+8y=40$ which contains the origin.

Thus the solution set of the system of inequalities is the intersection of these solution sets and is exhibited in the figure by the shaded portion which is common to the solution sets of all the individual inequalities.

We may observe that the solution set is the set of all points lying within and on the polygon with vertices $(0, \frac{5}{3})$, $(0, 5)$, $(4, 0)$, $(\frac{40}{3}, 0)$.

Note. It is interesting to observe that the solution set of any finite system of linear inequalities (with $\leqslant$ or $\geqslant$ sign only) will always be one of the following forms only :

(*i*) the solution set is the set of all points within and on some polygon with a finite number of vertices ;

(*ii*) the solution set contains a finite number of vertices, but is open on one side ;

(*iii*) the solution set is empty.

The following theorems will help the reader arrive at these results.

Theorem I. *If* (x_1, y_1) *and* (x_2, y_2) *belong to the solution set of the inequality*

$$ax+by \leqslant c,$$

then $\{tx_1+(1-t)x_2,\ ty_1+(1-t)y_2\}$ *also belongs to the solution set of the inequality for any value of* t, *such that* $0 \leqslant t \leqslant 1$.

Proof. Since (x_1, y_1) and (x_2, y_2) belong to the solution set, therefore

$$ax_1+by_1 \leqslant c \qquad \ldots (i)$$

$$ax_2+by_2 \leqslant c \qquad \ldots (ii)$$

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Let R be *any* point on the line segment PQ. Let its co-ordinates be (x_0, y_0).

Assume that

$$\frac{RQ}{PQ}=t. \qquad \dots \quad (i)$$

Now, triangles RQT and PCQ are similar. Therefore,

$$\frac{RQ}{PQ}=\frac{TQ}{CQ}=\frac{x_2-x_0}{x_2-x_1}. \qquad \dots \quad (ii)$$

From (*i*) and (*ii*), we get

$$\frac{x_2-x_0}{x_2-x_1}=t$$

or $$x_0=tx_1+(1-t)x_2.$$

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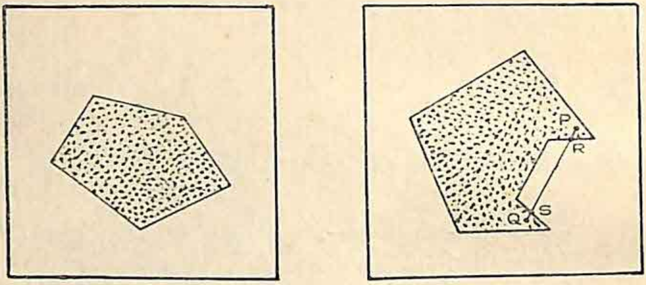
In a similar way, by drawing the perpendiculars on the y-axis, we can prove that $y_0=ty_1+(1-t)y_2$,

It may be noted that since R lies on the finite line segment PQ, therefore $0 \leqslant \frac{RQ}{PQ} \leqslant 1$, *i.e.*, $0 \leqslant t \leqslant 1$.

Note. In view of the note after theorem 1 and the theorem 2, we may now state the following result: If (x_1, y_1) and (x_2, y_2) belong to the solution set, A, of any finite system of linear inequalities, then all the points on the line segment joining (x_1, y_1) and (x_2, y_2) belong to the solution set A.

5. Convex Set. Def. A set, S say, of points in a plane is said to be convex if for any two points (x_1, y_1) and (x_2, y_2) belonging to the set S, all the points on the line segment joining (x_1, y_1) and (x_2, y_2) belong to the set S.

As examples, consider the two sets of points as shown in fig. 1.12.



Set A Set B

Fig. 1.12.

Set A is convex whereas set B is not, because a portion RS of the line segment PQ does not belong to the set B.

Using this definition, we can now make the following assertion :

The solution set of a finite system of linear inequalities in two unknowns is one of the following forms only :

(*i*) closed convex polygon,
(*ii*) open convex polygon,
(*iii*) empty set.

EXERCISES

1. Find graphically the solution sets of each of the following systems of linear equations :

(*i*) $4x+5y=6$;
$3x+8y=9$.

(*ii*) $10x+3y=18$;
$9x+16y=14$.

(*iii*) $7x+9y=3$;
$7x+9y=6$.

(*iv*) $8x+10y=7$;
$16x+20y=14$.

(*v*) $5x+8y=7$;
$7x+10y=13$.
$7.5x+12y=10{\cdot}5$.

(*vi*) $6x+7y=8$;
$8x+9y=7$.
$4x+5y=6$.

(*vii*) $2x+5y=6$;
$2x+5y=7$;
$2x+5y=10$.

2. Find grahpically the solution sets of each of the following systems of linear inequalities :

(*i*) $3x+4y \leq 6$;
$5x+9y \leq 67$;
$x \geq 0$;
$y \geq 0$.

(*ii*) $4x+5y \geq 8$;
$7x+8y \leq 9$.

(*iii*) $x+y \geq 1$;
$x \geq 1$;
$y \geq 1$.

(*iv*) $x-y \leq 1$;
$x-y \leq 4$;
$y \geq 0$.

(*v*) $x+y \leq 6$;
$x-y \leq 1$;
$2x+y \geq 6$;
$0{\cdot}5x-y \geq -4$;
$x \geq 1$;
$y \geq 0$.

(*vi*) $x+y \geq 2$;
$x-y \geq -2$;
$x+y \leq 6$;
$x-3y \leq 2$;
$x \geq 0$;
$y \geq 0$.

(*vii*) $2x-y \geq 0$;
$0{\cdot}5x-y \leq 0$;
$x+y \leq 5$;
$x \geq 0$;
$y \geq 0$.
x is an integer.

(*viii*) $x+y \leq 6$.
$2x+5y \geq -8$;
$3x+8y \leq 9$;
$10x+12y \leq 7$;
$8x+7y \geq 14$;
$x \geq 0$;
$y \geq 0$.

3. A convex polygon has the points $(-1, 0)$, $(3, 4)$. $(0, -3)$ and $(1, 6)$ as its vertices. Find a set of inequalities which defines the convex polygon having these vertices.

4. A man is considering to buy shirts at a sale : dress shirts at Rs. 20 each, sports shirts at Rs. 15 each. His budget permits spending at most Rs. 120 on the shirts, and since he needs dress shirts more than sports shirts, he decides to spend at least twice as much on the former as on the latter kind. He finds to his taste only 5 dress shirts and sports shirts. How many of each kind of shirt can he buy under all these conditions?

5. Assume that the minimal requirements of human beings are given by the following table :

	Phosphorus	*Calcium*
Adult	·02	·01
Child	·03	·03
Infant	·01	·02

Plot the amount of phosphorus on the vertical axis and the amount of calcium on the horizontal axis. Then draw in the convex sets of minimal diet requirements for adults, children (non-infants), and infants. State whether or not the following assertions are true :

(*a*) If a child's needs are satisfied, so are an adult's.

(*b*) An infant's needs are satisfied only if a child's needs are.

(*c*) An adult's needs are satisfied only if an infant's needs are.

(*d*) Both an adult's and an infant's needs are satisfied only if a child's needs are.

(*e*) It is possible to satisfy adult's needs without satisfying the needs of an infant.

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2

Fundamental Theorem

The Solution Procedures

In the previous chapter we learnt that the solution set of a system of finite number of linear equations and inequations in two unknowns is one of the following subsets of points of the plane :

(*i*) a closed convex polygon ;
(*ii*) an open convex polygon ;
(*iii*) an empty set of points.

In the present chapter, we propose to tackle the main problem of this book, namely, to find out the point or points of a closed convex polygon where the linear expression $ax+by$ assumes its maximum (or minimum). To be more precise, we will prove the following *fundamental theorem* of linear programming for two variables only : The expression $ax+by$ defined over a non-empty closed convex polygon assumes its maximum (minimum) value at atleast one vertex of the polygon.

We break the proof of this theorem in three steps, which we state and prove below in the form of theorems.

Theorem A. *If values of the expression* $ax+by$ *are considered over the set of points lying on the finite line segment joining the points* (x_1, y_1) *and* (x_2, y_2), *then the minimum (maximum) of*

$ax+by$ occurs at the end points of the line segment at least i.e. at (x_1, y_1) or at (x_2, y_2) at least.

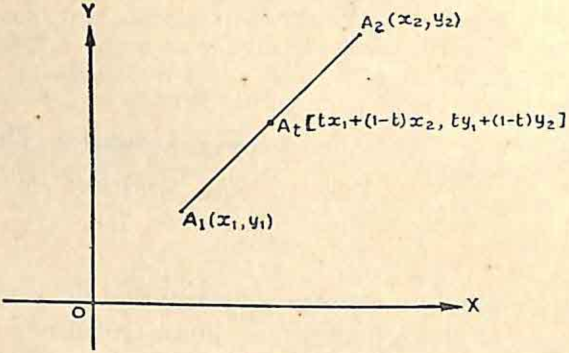


Fig. 2.1.

Proof : Let A_1 be the point (x_1, y_1) and A_2 be the point (x_2, y_2). By theorem 2 of the previous chapter, we know that the co-ordinates of any point on the line segment joining A_1 and A_2 are of the form $[t\,x_1+(1-t)x_2,\ ty_1+(1-t)y_2]$ where $0 \leqslant t \leqslant 1$.

Let A_t denote this general point. Also, let m_1, m_2 and m_t denote the values of $ax+by$ at A_1, A_2 and A_t respectively, so that

$$m_1=ax_1+by_1$$

$$m_2=ax_2+by_2$$

and

$$m_t=a[tx_1+(1-t)x_2]+b[ty_1+(1-t)y_2]$$

$$=m_2+t(m_1-m_2). \qquad \ldots \quad (A)$$

Three possibilities are to be considered now, namely, (*i*) $m_1=m_2$, (*ii*) $m_1<m_2$, (*iii*) $m_2<m_1$. We consider the first two and leave the third as an exercise.

Case (*i*) Let $m_1=m_2$. Then, from equation (*A*)

$$m_t=m_2+0=m_2 \text{ for all } t.$$

Hence in this case, the expression $ax+by$ assumes the same constant value m_2 (or m_1) for all points lying on the line segment $A_1\ A_2$. Therefore, the maximum as also the minimum values of $ax+by$ occur, in particular, at A_2 (or A_1).

Case (*ii*) Let $m_1<m_2$. We will prove that

$$m_1 \leqslant m_t \leqslant m_2$$

for all values of t such that $0\leqslant t\leqslant 1$. Thus in this case m_1 will be the minimum value and m_2 the maximum value of $ax+by$.

We have

$$\begin{aligned} m_1-m_t &= m_1-m_2-t(m_1-m_2) \\ &= (m_1-m_2)(1-t) \\ &\leqslant 0 \quad (\because\ m_1-m_2<0 \text{ and } 1-t\geqslant 0 \text{ for } 0\leqslant t\leqslant 1) \end{aligned}$$

$$\therefore \qquad m_1\leqslant m_t \text{ for } 0\leqslant t\leqslant 1 \qquad \ldots \quad (1)$$

Similarly,

$$\begin{aligned} m_t-m_2 &= m_2+t(m_1-m_2)-m_2 \\ &= t(m_1-m_2) \\ &\leqslant 0 \quad (\because\ m_1-m_2<0 \text{ and } t\geqslant 0) \end{aligned}$$

$$\therefore \qquad m_t\leqslant m_2 \text{ for } t\geqslant 0 \qquad \ldots \quad (2)$$

From (1) and (2), we find that

$$m_1\leqslant m_t\leqslant m_2 \text{ for } 0\leqslant t\leqslant 1.$$

Case (*iii*) can be similarly disposed of.

Thus, we can now state the following results :

In case (*i*), the expression $ax+by$ assumes its maximum as also the minimum values at any point on the line segment $A_1\ A_2$.

In case (*ii*), the expression $ax+by$ assumes its minimum value at A_1 and maximum value at A_2.

In case (*iii*), the expression $ax+by$ assumes its minimum value at A_2 and maximum value at A_1.

This completes the proof of the theorem.

Note. We may prove the above results graphically as shown in fig. 2.2. We have seen that

$$m_t=m_2+t(m_1-m_2)$$

The graph of m_t against t can be plotted as in fig. 2.2.

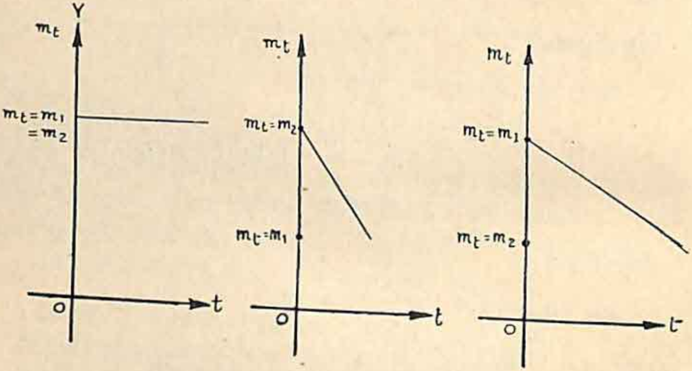


Fig. 2.2.

These graphs are drawn under the assumption that m_t is positive for $0 \leqslant t \leqslant 1$. Similarly, the graphs can be drawn when m_t assumes negative values also. These graphs now indicate that m_t always lies between m_1 and m_2 for $0 \leqslant t \leqslant 1$. Hence the result of the theorem.

Theorem B : *If the values of the expression $ax+by$ are considered over the set of points lying on the boundary of a non-empty closed polygon, the minimum (maximum) of $ax+by$ occurs on at least one of the vertices of the polygon.*

Proof. Let us assume that we are considering the minimum of $ax+by$. The case when $ax+by$ is to be maximised can be similarly discussed.

Since the polygon is closed, it has a finite number of boundary *line segments*. Consider one of these boundary line segments. By theorem A, the minimum of $ax+by$ when considered over this line segment, occurs at one of the end points of the line segment. Let it be A_1. Take up the second boundary line segment and mark the end point where $ax+by$ assumes its minimum value over this line segment. Exhaust all the boundary line segments this way. We are then left with a *finite* set of points, which are vertices of the polygon, where the expression $ax+by$ can possibly assume its minimum value. Now, out of these points pick up the point where $ax+by$ assumes its minimum value. This is the required point where $ax+by$ assumes its minimum value over the entire boundary of the polygon. Obviously, this is one of the vertices of the polygon.

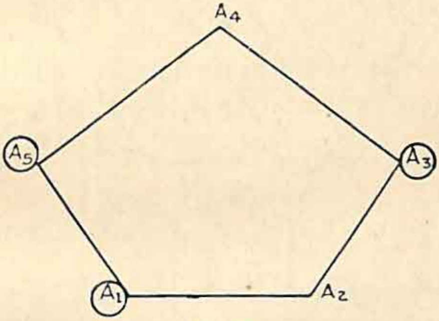


Fig. 2.3.

Theorem C : (*The Fundamental Theorem*). *If the values of the expression $ax+by$ are considered over the set of points constituting a non-empty closed convex polygon, then the minimum* *(maximum) of $ax+by$ occurs on at least one of the vertices of the polygon.*

Proof. Once again, we will attend to the problem of minimisation. We prove first that the minimum of $ax+by$ cannot occur at a point, P say, which lies inside the polygon.

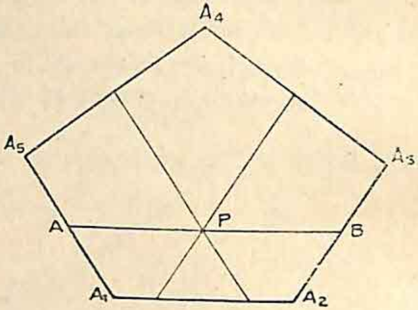


Fig. 2.4.

Let P be a point that lies inside the polygon. Since the polygon is closed, any line through P will intersect the boundary of the polygon in exactly two points, say A and B. By theorem A, the value of $ax+by$ is less (at most equal to) either at A or at B when compared with its value at P. Thus, for any point P, which lies inside the polygon, we can find at least one point, say B, on the boundary of the polygon such that the value at P of $ax+by$ is greater than (at most equal to) the value at B of $ax+by$. Hence, at no point, which lies inside the polygon, the expression $ax+by$ can assume its minimum value.

Consequently, the minimum value of $ax+by$ occurs at some point which lies on the boundary of the polygon. Now by making an appeal to theorem B, we prove the theorem at hand.

This completes the proof of the fundamental theorem of linear programming for two variables.

Examples

Example I. Maximise the expression $8x+9y$ subject to the restrictions

$$\begin{aligned} 2x+3y &\leqslant 6, \\ 3x-2y &\leqslant 6, \\ y &\leqslant 1, \\ x &\geqslant 0, \\ y &\geqslant 0. \end{aligned}$$

Solution. The solution set of the restrictions is obtained by using the methods explained in chapter 1 and is shown in fig. 2.5 graphically :

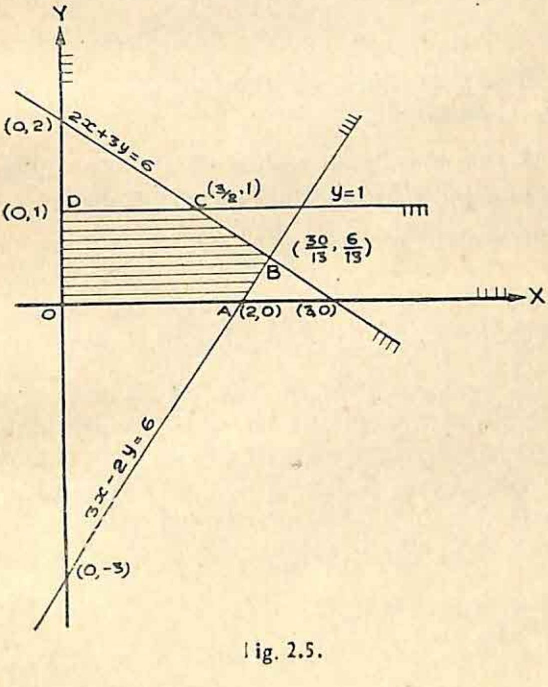


Fig. 2.5.

The solution set is the set of all points lying on and within the closed convex polygon *OABCD*.

Now, we know by the fundamental theorem that the maximum value of the expression $8x+9y$ which is to be considered over the closed convex polygon *OABCD*, occurs on at least some vertex of the polygon. Let us, therefore, compute its value at each of the vertices and then pick up the vertex at which the given expression is maximum.

Value of $8x+9y$ at $O(0, 0)=0+0=0$
,, ,, ,, ,, $A(2, 0)=8 \cdot 2+9 \cdot 0=16$
,, ,, ,, ,, $B(\frac{30}{13}, \frac{6}{13})=8 \cdot \frac{30}{13}+9 \cdot \frac{6}{13}=22\frac{7}{13}$
,, ,, ,, ,, $C\ (\frac{3}{2}, 1)=8 \cdot \frac{3}{2}+9 \cdot 1=21$
,, ,, ,, ,, $D\ (0, 1)=8 \cdot 0+9 \cdot 1=9$

Out of these five values, we find that the value of $8x+9y$ at $B(\frac{30}{13}, \frac{6}{13})$ is maximum.

Thus the maximum value of $8x+9y$ subject to the given restrictions is $22\frac{7}{13}$ and is attained when

$$x=\frac{30}{13},\ y=\frac{6}{13}.$$

Example 2. Compute the maximum value of the expression $2x+7y$ subject to the restrictions given in example 1.

Solution. As before, one can easily compute the value of $2x+7y$ at the five points O, A, B, C, D and pick up the maximum value. In fact, these values are 0, 4, $\frac{102}{13}$, 10, and 7 at the vertices O, A, B, C and D respectively. Thus, the maximum value of $2x+7y$ considered over the given polygon is 10 and is attained at the vertex $C\ (\frac{3}{2}, 1)$.

The purpose, however, of this example is to illustrate another solution procedure which does not make use of the fundamental theorem explicitly. To understand this procedure clearly, we recall the solution set of the restrictions as shown in fig. 2.6.

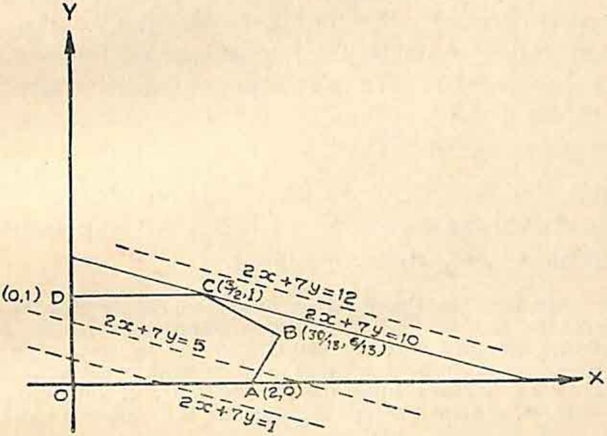


Fig. 2.6.

Our problem, as we know, is to find out a point lying on or within the polygon *OABCD* at which the expression $2x+7y$ is maximum.

Let us plot the line $2x+7y=1$, say (here, 1 on the right-hand side has been chosen arbitrarily). For all points on this line the expression $2x+7y$ has value 1, and we also note that the line $2x+7y=1$ has a line segment in common with the polygon *OABCD*. Similarly, if we plot the line $2x+7y=5$, we find that the line $2x+7y=5$ still has a line segment in common with the polygon *OABCD*. Also, as is obvious, the lines $2x+7y=1$ and $2x+7y=5$ are parallel.

This observation gives us a hint as to what should we do so as to maximise the expression $2x+7y$ over the polygon *OABCD*. The hint is the following :

If we draw the parallel lines $2x+7y=a$ for larger and larger values of a, then we are moving towards the point C.

We now continue to give larger and larger value to a till the line $2x+7y=a$ has at least one point common with the polygon $OABCD$. We find that the line $2x+7y=a$ has only one point, namely C, in common with the polygon $OABCD$ when $a=10$ We may also convince ourselves that if $a > 10$, then the line $2x+7y=a$ has no point in common with the polygon $OABCD$.

Thus the maximum value of the expression $2x+7y$ considered over the polygon $OABCD$ is 10 and is attained at the point C, *i e.*, when $x=\frac{3}{2}$, $y=1$.

Note. We hope that the reader can very well decide now the relative efficiencies of the two solution procedures. While applying the first procedure no ruler is needed at all to get the correct answer, whereas one may sometimes be in trouble without a ruler while applying the second procedure. We may also say that the first procedure is completely mathematical in nature, whereas the second procedure is intuitive in nature.

Example 3. Maximise the expression $4x+6y$ subject to the restrictions given in example 1.

Solution. We find that the values of $4x+6y$ evaluated at O, A, B, C, and D are 0, 8, 12, 12, and 6 respectively.

Thus, the maximum value of $4x+6y$ when considered over the polygon $OABCD$ is 12. But, we also observe an additional fact here : namely, the value of $4x+6y$ is 12 at B as also at C. Also, if we plot the line $4x+6y=12$, then we find that this line has a complete line segment, BC, in common with the polygon $OABCD$. Thus, if we substitute the values of x and y as specified by the co-ordinates of any point on the line segment BC, then the value of the expression $4x+6y$ is 12.

Hence, we may now say that the maximum value of the expression $4x+6y$ is 12 and that this maximum value is attained at an infinite number of points of the polygon $OABCD$; namely, any point on the line segment BC.

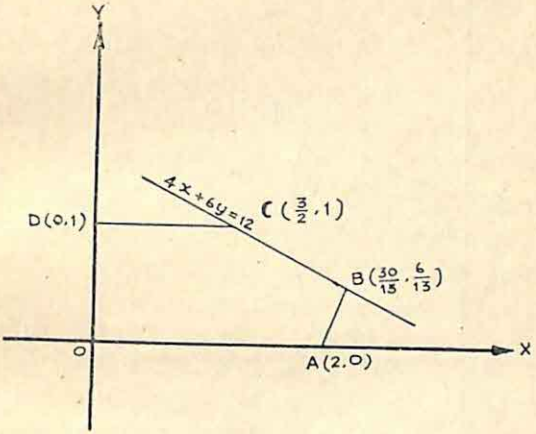


Fig. 2.7.

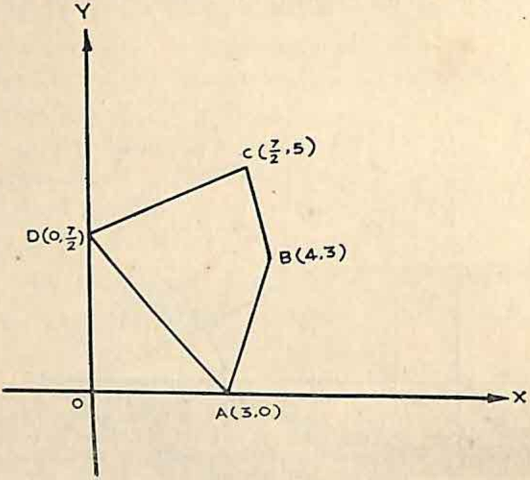
Note 1. Can you guess the reason as to why the maximum value of the expression $4x+6y$ is attained at an infinite number of points of the solution set of the restrictions. If you fail to guess, have a look once again at the restrictions and the solution procedure outlined in example 2. If you still fail, refer to exercise 6.

Note 2. Thus far the examples that have been considered, are only illustrations of the fundamental theorem of linear programming. We now present some examples which ensure that the polygon over which an expression like $ax+by$ is sought to be maximised (minimised) must be convex, closed and non-empty if the theorem is to hold good.

Example 4. Assuming that x and y can be natural numbers only, find out the point or points of the following polygon $ABCD$, next page (fig. 2.8) where the expression $2x+y$ is maximum.

Solution. Since now x and y can be natural numbers only, the expression $2x+y$ needs to be considered only for those points lying on or within the polygon $ABCD$ which have natural number co-ordinates. Thus, we have to consider only the circled points as shown in fig. 2.9 (scale has been changed).

Since now the expression $2x+y$ is to be considered over a finite number of points (whose coordinants are given



Fjg. 2.8.

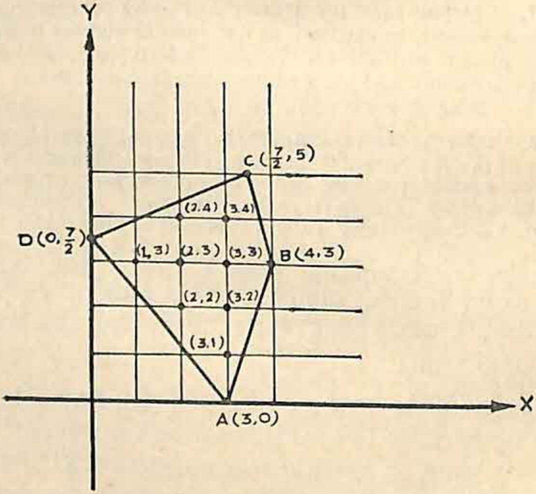


Fig. 2.9.

fig. 2.9) we find that the requirements of the fundamental theorem of linear programming are not fulfilled. We, therefore, do not have any reason to believe that the assertion of the theorem. *i.e.*, the maximum (minimum) of the given expression lie, on at least one vertex of the polygon, is valid.

However, with a little application of intuition, such problems can also be solved by taking full advantage of the assertion of the theorem. For this, we proceed as follows :

Let us consider the following question : Can we find out a closed convex polygon lying completely within the given polygon so that all the vertices of the new polygon have natural numbers as their co-ordinates and that all points with natural number co-ordinates in the original polygon are included in the new polygon ? The answer to this question, speaking intuitively, is 'yes'. In fact we can mark all the points with natural number co-ordinates nearest to the bundary of the original polygon and join the points thus obtained by means of line segments. This will result in the new polygon that is sought in the question raised above.

Having obtained such a new closed convex polygon, we can apply the fundamental theorem and obtain the vertex of the new polygon where the given expression assumes its maximum (or minimum). This vertex will have natural numbers as its co-ordinates (recall the construction of the new polygon). This point is then the required point where the given expression assumes its maximum (or minimum). (Why ?)

Applying this procedure to the problem at hand, we see that the new polygon is as shown in fig. 2.10.

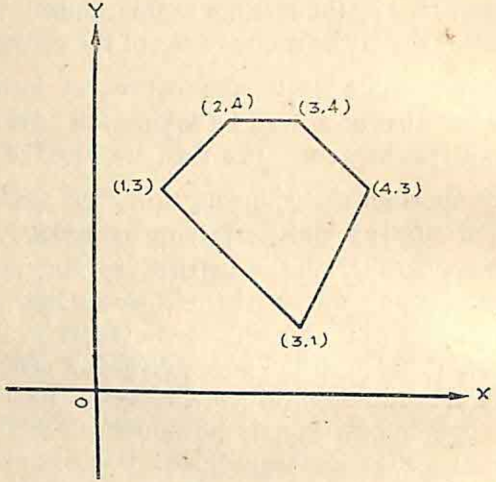


Fig. 2.10.

The expression $2x+y$ assumes its maximum value at the point (4, 3) and the maximum value is $2\times 4+3=11$.

Note 1. It may be observed that if the restriction 'the co-ordinates are to be natural numbers' is relaxed, then the expression $2x+y$ assumes its maximum value at $C(\frac{7}{2}, 5)$. One may, therefore, be inclined to say that if the restriction about natural numbers is insisted upon, then the required point will be the one with natural number co-ordinates lying within the given polygon which is *nearest* to the point $(\frac{7}{2}, 5)$. The example shows that this is not necessary.

Note 2. The number of points lying within or on a closed convex polygon with natural numbers as their co-ordinates is always finite. So one may take the attitude that the given expression can be evaluated at each of these finite number of points and then that point picked up which yields the maximum (or minimum) value of the given expression. True, this could be done if the total number of such points is not very large. In any case, we believe that the procedure that we have outlined above will surely save a good deal of computational effort and it also helps us in appreciating the fundamental theorem.

Example 5. Maximise the expression $4x+5y$ subject to the restrictions

$$\begin{aligned} x+y &\geqslant 1, \\ y-2x &\leqslant 1, \\ 4x-2y &\leqslant 1, \\ x &\geqslant 0, y \geqslant 0. \end{aligned}$$

Solution. The solution set of the restrictions is shaded as shown in fig. 2.11.

We find in this case that the solution set is convex but not closed. Therefore, the fundamental theorem cannot be applied here.

However, if we apply the procedure that was outlined in example 2, then the line $4x+5y=a$ has a complete line

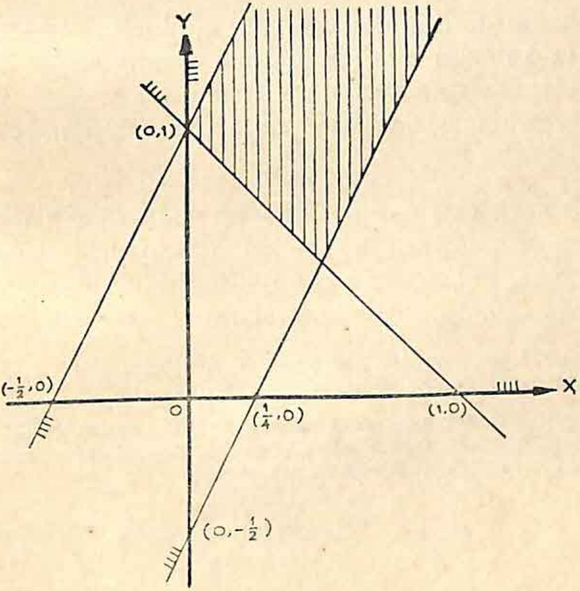


Fig. 2.11.

segment in common with the solution set for *any large* value of a. Thus, if all points in the solution set are to be considered, then the maximum of $4x+5y$ over this particular solution set does not exist. In such a case we say that the linear programming problem has an *unbounded* solution. Physically, if $4x+5y$ measures some profit, then the existence of an unbounded solution implies that an *unlimited* profit can be realised while satisfying all the restrictions that are *considered.* In practice, however, we know that an unlimited profit can never be realised and, therefore, the existence of an unbounded solution ensures that some *important* restriction has been overlooked.

Example 6. Maximise the expression $4y-8x$ subject to the restrictions of example 5.

Solution. In this case we find that the line $4y-8x=a$ has posits in common with the solution set only when $-2 \leqslant a \leqslant 4$. Thus, there exists a maximum value of a *i.e.*, of $4y-8x$ when the expression is considered over the solution set of the restrictions.

Also this maximum value 4 is achieved at an infinite number of points of the solution set, namely the set of all points lying in the first quadrant on the line passing through the point $A(0, 1)$. (See fig. 2.11).

Note 1. The reader is advised to compare this example with example 3 and find out the similarities and dissimilarities.

Note 2. It may sometimes happen that the solution set of the restrictions is empty. In such a case we say that the maximum and minimum values of the expression $ax+by$ **do not exist.**

EXERCISES

1. Solve each of the following linear programming problems where $x \geqslant 0$, $y \geqslant 0$:

(*i*) maximise $x+3y$ subject to

$$3x+6y \leqslant 8$$
$$5x+2y \leqslant 10.$$

(*ii*) maximise $6x+15y$ subject to

$$5x+3y \leqslant 15$$
$$2x+5y \leqslant 10.$$

(*iii*) maximise $5x+24y$ subject to

$$3x+8y \leqslant 24$$
$$x+y \geqslant 2$$
$$x \geqslant 3$$
$$y \leqslant 23.$$

(*iv*) minimise $3x+5y$ subject to

$$-3x+4y \leqslant 12$$
$$2x-y \geqslant -2$$
$$2x+3y \geqslant 12$$
$$x \leqslant 4$$
$$y \geqslant 2.$$

(*v*) maximise $3x+5y$ subject to

$$2x-3y \geqslant 6$$
$$9x+4y \geqslant 12.$$

(*vi*) maximise $4x-3y$ subject to

$$3x-2y \leqslant 6$$
$$-6x+4y \leqslant -8.$$

(*vii*) maximise $3x+8y$ subject to

$$3x-4y \geqslant 0$$
$$2x-5y \geqslant 15.$$

(*viii*) maximise $8x+9y$ subject to

$$3x+4y \leqslant 0$$
$$4x+5y \geqslant 0$$
$$2x+9y \leqslant 0.$$

(*ix*) maximise $4x+\cdot 3y$ subject to

$$\begin{aligned} x+y &\leqslant 8000\\ 2x+y &\leqslant 1000\\ x &\leqslant 400\\ y &\leqslant 700. \end{aligned}$$

(*x*) maximise $3x+2y$ subject to

$$\begin{aligned} x+y &\geqslant 1\\ x &\geqslant y\\ x &\leqslant 1. \end{aligned}$$

2. Find in each of the following cases the maximum and the minimum values of expression $x-y$ where $x\geqslant 0$, $y\geqslant 0$:

(*i*)
$$\begin{aligned} -2x+y &\leqslant 2\\ x-2y &\leqslant 2\\ x+y &\leqslant 5. \end{aligned}$$

(*ii*)
$$\begin{aligned} -2x+y &\leqslant -2\\ x-2y &\leqslant 2\\ x+y &\leqslant 5. \end{aligned}$$

(*iii*)
$$\begin{aligned} -2x+y &\leqslant 2\\ -x+2y &\geqslant 0\\ x+y &\leqslant 5. \end{aligned}$$

(*iv*)
$$\begin{aligned} 2x-y &\leqslant 4\\ x-2y &\leqslant 2\\ x+y &\leqslant 5. \end{aligned}$$

(*v*)
$$\begin{aligned} -2x+y &\leqslant 2\\ x-y &\leqslant 2\\ x+y &\leqslant 5. \end{aligned}$$

3. Solve each of the problems given in exercises 1 and 2 when x and y are further restricted to be natural numbers.

4. Consider the following problem :
maximise $2x+7y$ subject to

$$\begin{aligned} 5x+8y &< 18\\ 6x+7y &< 12\\ x &\geqslant 0\\ y &\geqslant 0. \end{aligned}$$

Do you think that you can apply the fundamental theorem in this case ? If not, why ?

Try to make as precise a statement as you can about the maximum value of $2x+7y$.

5. In example 5, a physical interpretation was given for the unbounded solution. Try to give a physical interpretation for the situations described in examples 3 and 6.

6. Consider example 3 of the text and observe that the expression $4x+6y$ when equated to any constant value is actually parallel to one of the restrictions of the problem.

Prove that any linear programming problem which behaves as does example 3 of the text must satisfy the observation mentioned above.

Do you think that the same is true if the linear programming problem behaves as example 6 of the text ?

7. If the expression $(ax+by+\alpha)/(cx+dy+\beta)$ is defined over a set of points constituting a closed convex polygon, prove that its minimum (and maximum) value occurs on at least one vertex of the polygon.

Try to find out some other forms of the expression to be maximised (minimised) so that the above property is true.

8. Consider the polygonal convex set P defined by the inequalities

$$-1 \leqslant x \leqslant 4,$$
$$0 \leqslant y \leqslant 6.$$

(*i*) Find four different sets of conditions on the constants a and b so that the expression $ax+by$ should have its maximum at one and only one of the four vertices of P.

(*ii*) Find conditions so that $ax+by$ shouldhave its minimum at each of these points.

(*iii*) Find conditions so that $ax+by$ should have its maximum on at least two of the vertices.

9. Consider the following pair of problems :

(*i*) maximise d_1x+d_2y subject to

$$a_1x+b_1y\leqslant c_1$$
$$a_2x+b_2y\leqslant c_2$$
$$x\geqslant 0,\ y\geqslant 0.$$

(*ii*) minimise c_1x+c_2y subject to

$$a_1x+a_2y\geqslant d_1$$
$$b_1x+b_2y\geqslant d_2$$
$$x\geqslant 0,\ y\geqslant 0.$$

By giving any specific values to the constants a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 observe that the maximum value of d_1x+d_2y equals the minimum value of c_1x+c_2y.

Try to prove in general that whatever be the values of the constants involved in the above problems the maximum value of d_1x+d_2y always equals the minimum value of c_1x+c_2y.

(A pair of problems as given above is called a pair of *dual linear* programming problems.)

10. Prove that if the maximum of the expression $ax+by$ when considered over a closed convex polyhedron occurs at the point (x_1, y_1), then

(*i*) the miximum of $ax+by+\alpha$, where α is some fixed constant, also occurs at the point (x_1, y_1) ;

(*ii*) the minimum of $-(ax+by)$ also occurs at the point (x_1, y_1).

Also in case (*ii*) prove that

maximum value of $ax+by=-$[minimum value of $-(ax+by)$].

3

Applications of Linear Programming

During the past two decades it has been observed that various types of optimisation problems encountered in diverse spheres of life are, essentially speaking, linear programming problems. The purpose of the present chapter is to consider a few such problems.

The reader will do well to understand that the following steps are all essential whenever considering any application of mathematics :

(*a*) to recognise the existence of a problem in some sphere of life,
(*b*) to collect the essential data about the problem,
(*c*) to state the problem verbally in as precise terms as is possible,
(*d*) to formulate the verbal problem as a mathematical problem, if possible,
(*e*) to search for a mathematical solution procedure for the solution of the mathematical problem,
(*f*) to solve the problem, if a solution procedure is known.

In what follows we will consider a few practical problems which when formulated mathematically can easily be recognised as what we have called linear programming problems in the previous chapter. Since we already know now a solution procedure for such problems, we can easily solve these also.

Example I. Two types, A and B say, of a particular commodity are made on 2 machines, I and II say. To produce 1 kg. of type A, machines I and II must be used for 1 and $2\frac{1}{4}$ hours respectively. To produce 1 kg. of type B, machines I and II must be used 3 and $\frac{3}{4}$ hours respectively. No machine is allowed to operate more than 12 hours per day. Profits on type A and type B respectively are given to be Rs. 4 and Rs. 6 per kg. How many kg. of each should be made per day to maximise the profit ? What will the maximum profit be ?

Solution. Let x kg. of type A and y kg. of type B be made per day.

Total time required on machine I $=x+3y$

" " " " " II $=\frac{9}{4}x+\frac{3}{4}y$.

Profit per day =Rs. $(4x+6y)$.

Since, now each of the machines cannot be used for more than 12 hours a day, therefore x and y must satisfy

$$x+3y \leqslant 12 \qquad \ldots (1)$$

$$\frac{9}{4}x+\frac{3}{4}y \leqslant 12 \qquad \ldots (2)$$

Also, since negative quantities cannot be made, therefore,

$$x \geqslant 0 \qquad \ldots (3)$$

$$y \geqslant 0 \qquad \ldots (4)$$

Our problem, therefore, is to find out x and y which will yield the maximum profit, *i.e.*, which will maximise

$4x+6y$, when x and y are subject to the restrictions (1), (2), (3) and (4). This we easily recognise as a linear programming problem and, therefore, the solution procedures developed in the previous chapter are applicable.

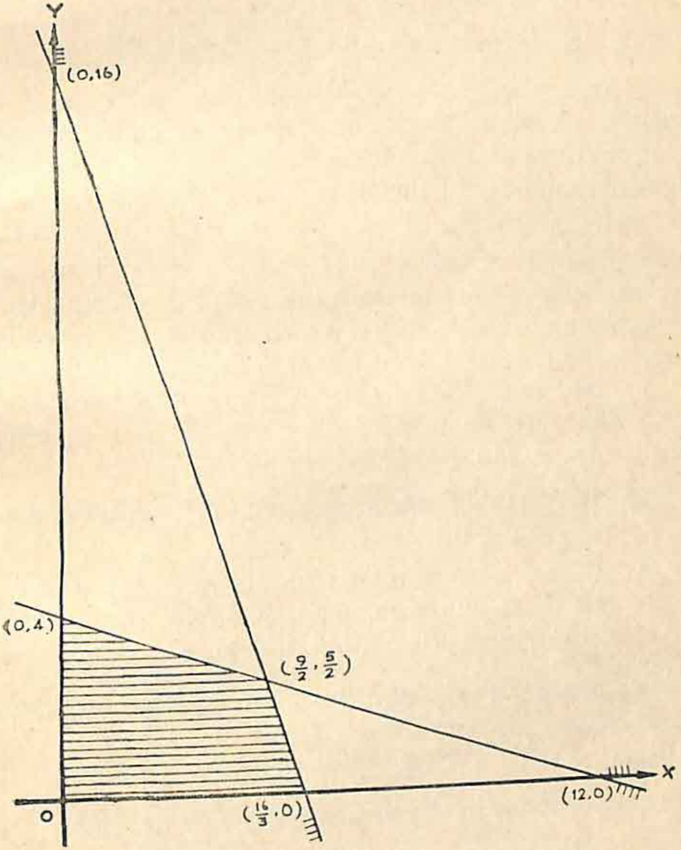


Fig. 3.1.

The solution set of the inequalities (1), (2), (3) and (4) is the shaded portion shown in fig. 3.1.

The maximum of $4x+6y$ occurs at the vertex $\left(\frac{9}{2}, \frac{5}{2}\right)$.

Thus the maximum profit is achieved when

$$x = 4{\cdot}5,$$
$$y = 2{\cdot}5,$$

and, maximum profit$=$Rs. $(4\times4{\cdot}5+6\times2{\cdot}5)$

$=$Rs. 33.

Example 2. A wholesale dealer deals in two kinds, A and B say, of mixtures of nuts. Each kg of mixture A contains 60 gms of almonds, 30 gms of cashew nuts, and 30 gms of hazel nuts. Each kg of mixture B contains 30 gms of almonds, 60 gms of cashew nuts, and 180 gms of hazel nuts. The remainder of both mixtures is peanuts. The dealer is contemplating to use mixtures A and B to make a bagwhich will contain at least 240 gms of almonds, 300 gms of cashew nuts, and 540 gms of hazel nuts. Mixture A costs Rs. 8·00 per kg and mixture B costs Rs. 12·00 per kg. Assuming that mixtures A and B are nuiform :

(*a*) How many kgs of each should he use to minimise the cost of the bag ?

(*b*) If there is no market for nut mixtures priced at Rs. 15 per kilogram, will you advise the dealer to introduce the type of mixture he is contemplating ?

Solution : Let x kgs of mixture A and y kgs of mixture B be used.

Then,

Total almonds in $(x+y)$ kgs $=(60x+30y)$ gms

,, cashewnuts ,, $=(30x+60y)$ gms

,, hazel nuts ,, $=(30x+180y)$ gms

Total cost of $(x+y)$ kgs $=$Rs. $(8x+12y)$.

According to the requirements of the dealer, we must have

$$\begin{aligned} 60x+30y &\geqslant 240\\ 30x+60y &\geqslant 300\\ 30x+180y &\geqslant 540\\ x &\geqslant 0\\ y &\geqslant 0. \end{aligned}$$

Thus our problem is now to find x and y which minimise $8x+12y$ where x and y are restricted by the above inequalities.

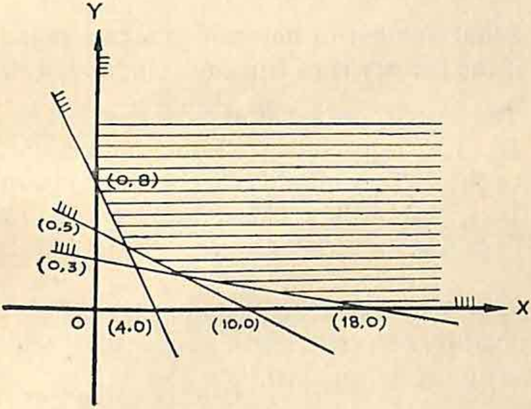


Fig. 3.2.

By using the solution procedure of the previous chapter we find that $8x+12y$ is *minimum* when $x=2, y=4$.

Thus to minimise the cost of the bag, the dealer should use 2 kgs of type A and 4 kgs of type B.

(*b*) The minimum cost of the bag $=$ Rs. $(8\times2+12\times4)$
$=$ Rs. 64.

Since the bag contains $4+2$ *i.e.*, 6 kgs of nuts, the minimum price per kg. of this new mixture is Rs. $\frac{64}{6}$ *i.e.*, Rs. 10·66.

Therefore, the dealer can safely introduce this type of mixture in the market.

Example 3. A factory makes cricket bats and tennis rackets. A cricket bat takes 1·5 hours of machine time and 2 hours of craftsman's time, while a tennis racket takes 2·5 hours of machine time and 1·5 hours of craftsman's time. In a day the factory has available up to 80 hours of machine time and 70 hours of craftsman's time.

(*a*) What is the largest number of (*i*) bats, (*ii*) rackets which could be made in a day in the factory ?

(*b*) What number of bats and rackets must be made if the factory is to work at full capacity ?

(*c*) The profit on a bat and on a racket is Rs. 5 and Rs. 3·50 respectively. Find the maximum profit to the factory on a day when it produces (*i*) only bats, (*ii*) only rackets, and (*iii*) works at full capacity.

Solution : Let x bats and y rackets be produced in a day. Then, as before, one can easily verify that the following inequations must be satisfied by x and y :

$$\begin{aligned} 1{\cdot}5x+2{\cdot}5y &\leqslant 80, \\ 2x+1{\cdot}5y &\leqslant 70, \\ x &\geqslant 0, \\ y &\geqslant 0. \end{aligned}$$

The solution set of these inequalities is shaded in fig. 3·3.

(*a*) (*i*) The largest number of bats that can be made in a day is obtained when we do not produce any racket that day, *i.e.* we have to look for the point in the solution set on the x-axis which has greatest x-co-ordinate. This point is (35, 0). Thus we can produce atmost 35 bats in a day.

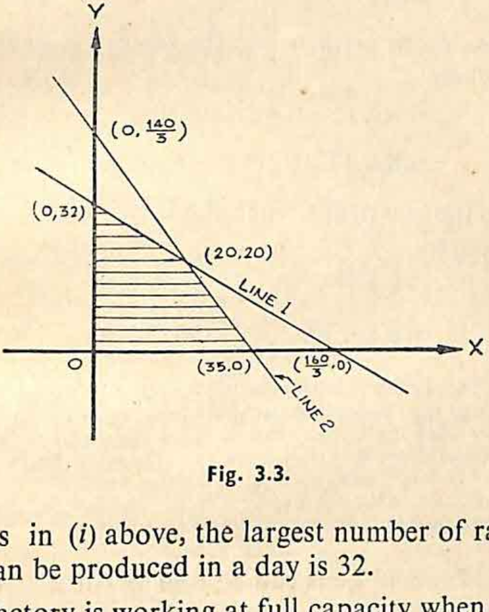


Fig. 3.3.

(*ii*) As in (*i*) above, the largest number of rackets that can be produced in a day is 32.

(*b*) The factory is working at full capacity when we do not waste any machine time and craftsman's time. For points lying *on* line 1 we are not wasting any machine time and for points lying on line 2 we are not wasting any craftsman's time. Therefore, at the point of intersection of these two lines, we are not wasting any time. We read from the figure that the point of intersection is (20, 20). Thus, if we produce 20 bats and 20 rackets, then the factory is working at full capacity.

(*c*) Total profit when we produce x bats and y rackets

$$= \text{Rs. } (5x + 3{\cdot}5y).$$

(*i*) Maximum profit when the factory is producing only bats

$$=\text{Rs. }(5\times 35+3{\cdot}5\times 0)$$
$$=\text{Rs. }175$$

(*ii*) Maximum profit when the factory is producing only rackets

$$=\text{Rs. }(5\times 0+3{\cdot}5\times 32)$$
$$=\text{Rs. }112.$$

(*iii*) Maximum profit when the factory is working at full capacity

$$=\text{Rs. }(5\times 20+3{\cdot}5\times 20)$$
$$=\text{Rs. }170$$

Note : This example illustrates the fact that if maximum profit is the only objective of the factory, then it should produce only bats. The reader may also note that the factory can make a larger profit even when the machine is not working at its full capacity. The time utilised on the machine is $1{\cdot}5\times 35=52{\cdot}5$ hours only *i.e.*, a wastage of 27·5 machine hours per day to realise the maximum profit. This is precisely because of the very large profit on one bat as compared with the profit on one racket.

Example 4. Old hens can be bought for Rs. 2·00 each but young ones cost Rs. 5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen costs Re. 1·00 per week to feed. If I have only Rs. 80·00 to spend for hens, how many of each kind should I buy to give a profit of more than Rs. 6·00 per week, assuming that I cannot house more than 20 hens ?

Solution. Let x old hens and y young hens be bought today. Then,

Total cost of old and young hens = Rs. $(2x+5y)$.

Therefore, x and y must satisfy the inequalities

$$\begin{aligned} 2x+5y &\leqslant 80, \\ x+y &\leqslant 20, \\ x &\geqslant 0, \\ y &\geqslant 0. \end{aligned}$$

Also x and y must be natural numbers.

No. of eggs laid by $x+y$ hens per week

$$=3x+5y$$

Revenue per week $=(3x+5y)$ 30 paise.

Expenditure per week $=$ Rs. $(x+y)$.

Profit per week $=$ Rs. $30\,\dfrac{(3x+5y)}{100}-$ Rs. $(x+y)$

$$= \text{Rs.}\left(\frac{5y-x}{10}\right)$$

Let us first of all find out the maximum profit that can be achieved in this bargain. The solution set of the restrictions is shown in fig. 3.4.

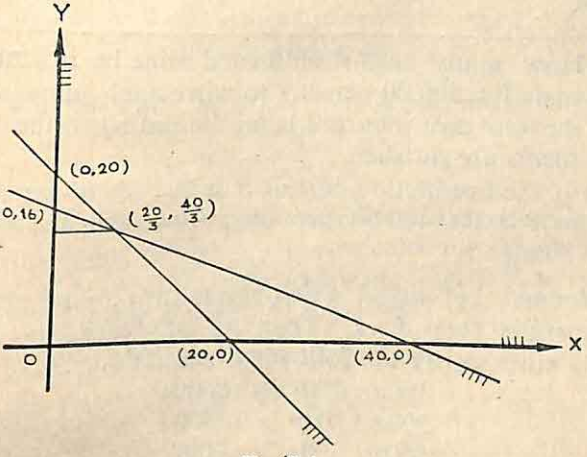


Fig. 3.4.

The maximum of $(5y-x)/10$ occurs at the point (0, 16) and the maximum profit is Rs. $\frac{5\times16-0}{10}=$ Rs. 8·00.

Thus one *possible solution* of the problem is to buy 16 young hens only. Try to find out other solutions, if any such exists. We claim that (2, 15) and (5, 14) are other solutions. Verify it. Also try to guess as to how we found these solutions.

Example 5. A company owns two coal mines, *A* and *B* say. The production in these mines and the total requirements are given in the table below :

	Production in A (quintals/day)	*Production in B (quintals/day)*	*Requirements (quintals)*
High grade	100	200	8000
Medium grade	300	200	15000
Low grade	500	200	20000

(*i*) How many days should each mine be operated if it costs Rs. 500·00 per day to work each mine so that the total cost incurred is minimised when the requirements are satisfied ?

(*ii*) If the production cost at *A* is Rs. 700·00 per day and at *B* is Rs. 400·00 per day, how will the solution change ?

Solution. Let mine *A* be operated for x days and mine *B* be operated for y days. Then, as may easily be argued, x and y must satisfy the following inequations :

$$\begin{aligned} 100x+200y &\geqslant 8000 \\ 300x+200y &\geqslant 15000 \\ 500x+200y &\geqslant 20000 \\ x \geqslant 0,\ y &\geqslant 0. \end{aligned}$$

The solution set of these inequations is as shown in fig. 3.5.

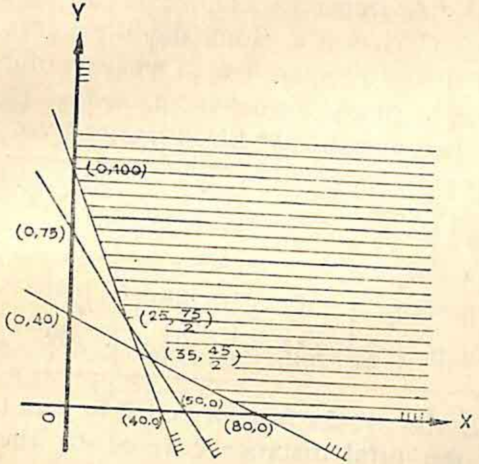


Fig. 3.5.

(*i*) Total cost of operating the mines

$$= \text{Rs. } 500\ (x+y).$$

We now find that 500 $(x+y)$ is minimum when

$$x=35,\ y=\frac{45}{2}.$$

Thus in order to fulfil the requirements with minimum cost, mines *A* and *B* should be operated for 35 and 22·5 days respectively. (We are assuming here that mines can be operated for any fraction of the day).

(*ii*) Total cost of operating the mines

$$= \text{Rs. } (700x+400y)$$

Now, $700x+400y$, over the feasible region is minimum when $x=25,\ y=37{\cdot}5.$

Thus, in this case mines A and B should be operated for 25 and 37·5 days respectively.

Example 6. A transport company has offices in five localities, A, B, C, D, and E. Some day the offices located at A and B had 10 and 15 spare trucks whereas offices at C, D and E required 8, 10 and 7 trucks respectively. The distances in kilometres between the five localities are given below :

From ↓ | *To*→	*C*	*D*	*E*
A	20	60	40
B	15	25	80

How should the trucks from A and B be sent to C, D and E so that the total distance covered by the trucks is minimum.

Solution. First we observe that the total number of trucks available at A and B is equal to the total number of trucks required at C, D and E.

Let x trucks be sent from A to C and y trucks be sent from A to D. Then by a simple argument now, we see that the number of trucks sent from A, B to C, D, E may be tabulated as follows :

Number of trucks to be sent *From* ↓ | *To*→	*C*	*D*	*E*
A	x	y	$10-(x+y)$
B	$8-x$	$10-y$	$15-[(8-x)+(10-y)]$ $=x+y-3$.

Also, we require that x and y satisfy the inequations :

$$\begin{aligned} x \geqslant 0,\ y &\geqslant 0 \\ 10-(x+y) &\geqslant 0 \\ 8-x &\geqslant 0 \\ 10-y &\geqslant 0 \\ x+y-3 &\geqslant 0 \end{aligned}$$

x and y are natural numbers.

With this schedule, the total distance covered

$$\begin{aligned} &= 20x+60y+40\,(10-x-y)+15\,(8-x)+ \\ &\qquad\qquad 25\,(10-y)+80(x+y-3) \\ &= 45x+75y+530. \end{aligned}$$

Now the solution set of the inequalities to be satisfied by x and y is shown in fig. 3·6.

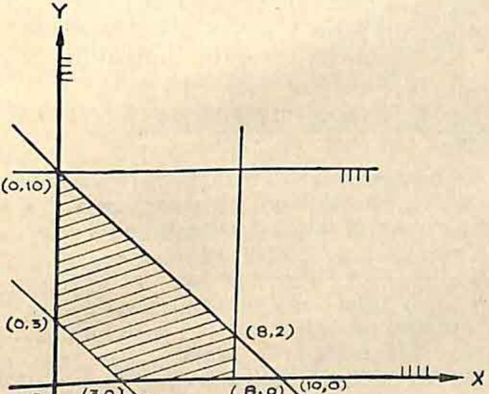


Fig. 3.6.

The minimum value of $45x+75y+530$ occurs at the point (3, 0), *i.e.*, $x=3$, $y=0$. Substituting these values of x and y

in the table that we made above, we find the required number of trucks to be sent from A and B to C, D and E. The table becomes :

Number of trucks to be sent *from* ↓ | *To* →	C	D	E
A	3	0	7
B	5	10	0

The total minimum distance covered is 665 kilometers.

EXERCISES

1. It is required to transport 500 troops and 42 tons of equipment to an African country. Two types of aircraft are available in sufficient numbers ; type 1 can carry 50 men and 5 tons of equipment, while type 2 can carry 40 men and 3 tons of equipment. For this journey, the operating cost for each type 1 aircraft is Rs. 1500 and for each type 2 aircraft is Rs. 1200. How many of each are required if the total cost is to be a minimum ?

2. A bicycle manufacturer makes two models, a sports cycle and a racing machine. In order to make a sports model, 6 man-hours are needed while a racing model requires 10 man-hours. (By 6 man-hours we mean either 1 man working 6 hours or 2 men working for 3 hours each, or 3 men working for 2 hours each, or 6 men working for 1 hour each). The manufacturer can employ no more than 15 men and these men work 8 hours per pay for 6 days a week. The cost of materials amounts to Rs. 50 per cycle and the manufacturer's total weekly quota of such materials may not exceed Rs. 4000. The firm has a contract to supply at least 30 sports models and 20 racers per week. How many cycles of each type should be made in order to obtain the maximum possible profit if the profit on each sports cycle is Rs. 10 and on each racing model it is Rs. 30.

3. A mine manager has contracts to supply 1000 tons of grade 1 coal, 700 tons of grade 2, 2,000 tons of grade 3, and 4,500 tons of grade 4. Seams *A* and *B* are being worked at a cost of Rs. 16,000 and Rs 40,000 respectively per shift, and the yields in ton per shift from each seam is given by the table below :

	Grade 1	*Grade 2*	*Grade 3*	*Grade 4*
Seam A	200	100	200	400
Seam B	100	100	500	1500.

How many shifts per week should each seam be worked in order to fulfil the contracts most economically.

4. A manufacturer makes two models, *A* and *B*, of a product. Each model must be processed by two machines. To complete one unit of model *A*, machine I must work 1 hour and machine II must work 2·5 hours. To complete one unit of model *B*, machines I and II must work 4 hours and 2 hours respectively. Machine I may not operate more than 8 hours per day, and machine II, not more than 12 hours per day. If the profit on model *A* is Rs. 20·00 per unit and on *B*, Rs. 28 00 per unit, how many units of each model should the manufacturer produce per day to maximise his profit ?

5. A furniture manufacturer makes two different types of card tables, the Deluxe and the Regal. Each type must be processed by three machines Alpha, Beeta and Gama. The times taken in hours by the models on the three machines, the times available in hours per week and the profits on the tables are given below :

	Alpha	*Beeta*	*Gama*	*Profit (Rupees/Table)*
Deluxe	2	3	4	20
Regal	3	4	2	23
Time Available	580	1000	840	

The manufacturer can sell all the tables of both types that he desires to produce. How many of each type should he produce every week to realise maximum profit ?

6. Two components P_1 and P_2 can be blended in two ways ; product A_1 contains 3 parts of P_1 to 1 part of P_2, while to make product A_2, 2 parts of P_1 are blended with 2 parts of P_2. The components P_1 and P_2 are available in quantities 100 quintals and 80 quintals per week respectively. The profits per quintal on products A_1 and A_2 are Rs. 10·00 and Rs. 15·40 respectively.

If $4x$ quintals of A_1 and $4y$ quintals of A_2 are made per week, show that $x+2y \leqslant 80$, and write down another inequality involving x and find out how much of each product should be made every week for maximum profit ?

7. In an airlift it is required to transport 1200 people and 90 quintals of baggage. Two types of planes are available ; type 1 can carry 60 passengers and 8 quintals of baggage, and type 2 can carry 85 passengers and 5 quintals of baggage. Only 20 type 1 and 15 type 2 planes are available. What is the smallest number of planes which can be used ?

8. A certain shop, having some machine time available, wished to schedule most profitably two types of tapered fastening pins. The problem was how many lots of each to schedule. Three machine tools were used : Lathe *A* for rough turn and cut off, Lathe *B* for finish turn and grinder *G* for finish grind. Spare time available on each machine was : A—50 hours, B—36 hours, and G—81 hours. The lot manufacturing times in hours are given below :

	Machine		
	A	*B*	*G*
Taper pin type 1	10	6	4·5
Taper pin type 2	3	6	18 0

Profit on these pins was Rs. 27·00 per lot for type 1 and Rs. 21·00 per lot for type 2 taper pins. How will the solution change if the management policy requires that at least one lot each of taper pin types 1 and 2 must be scheduled in the spare machine time available.

9. A certain manufacturer of screw fastenings found that there was a market for packages of mixed screw sizes. His market research data indicated that two mixtures of three screw types (1, 2 and 3) properly priced, could be most acceptable to the public. The relevant data is :

Mixture	*Specification*	*Selling price* (Rupees/kg.)
A	$\geqslant$ 50 percent type 1 $\leqslant$ 30 percent type 2 any quantity of type 3	5
B	$\geqslant$ 35 percent type 1 $\leqslant$ 45 percent type 2 any quantity of type 3	4

For these screws the plant capacity and manufacturing cost are given below :

Screw Type	*Plant Capacity* (Kg./day × 100)	*Manufacturing cost* (Rupees/kg.)
1	10	4·50
2	10	3·50
3	6	2·70

What production shall this manufacturer schedule for greatest profit assuming that he can sell all that he manufactures.

10. An architect estimates that the annual cost of heating a building is Rs. 20·00 for every square meter of window and Rs. 7·00 for every square metre of wall or roof. A local bye-law states that the area of the windows must be atleast ⅛th as much the area of the walls and roof. What is the largest surface area a building can have if its heating cost is not to exceed 120·00 per month ?

11. (*a*) A firm makes two products *M* and *N*. The total production capacity is at most 10 tons altogether per day. The firm has a permanent contract to supply at least 3 tons of M

per day to another firm and another permanent contract to supply at least 2 tons of N per day to yet another firm. Each ton of M requires 20 machine hours in production, each ton of N requires 20 machine hours in production and the number of men and machines available is such that no more than 420 machine hours can be worked per day. The profit on each ton of M is Rs. 200.00 and on each ton of N is Rs. 300.00. Assuming that the whole output can be sold at this rate of profit, how much M and how much N should the firm make daily in order to maximise the profit ?

(*b*) The firm now has scope, for some reason or other, to vary its daily production capacity in the range 10 to 19 tons, inclusive, and its available daily machine hours in the range 420 to 510 inclusive, (assuming integer values only in both cases) and that it is now desired to investigate for maximum profit all possible combinations of these different conditions ; furthermore, all results must be tabulated for instant reference so that planned production in varying circumstances within the limits set, may be achieved.

12. A factory manager has to decide how many of each of two types of machines to instal. The facts about each are as follows :—

	Output/week	*Factory Floor Space*	*Labour Needed* (per machine)
Machine A	300 units	500 sq. meters	10 men
Machine B	200 units	600 sq. meters	6 men

The factory has 5050 sq. meters of floor space, and only 80 skilled workers are available. Can an output of 2,400 units a week be reached ? If so, how many of each machine are required ?

13. Two products, A and B, each require three sequential departmental production operations in the course af manufacture. The first two operations are performed in departments w and x. The last may be done in departments y and z or by overtime in z (which is designated z' below). Hence there are three schedules of manufacture possible for each product, as shown in the table below. This table gives the unit production times for each department, the unit profit, and the time available

for production in each department. It is required that the most profitable schedule of manufacture be determined for products *A* and *B* :

Production Deptt	*Product A Schedule*			*Product B Schedule*			*Production hours available*
	I	*II*	*III*	*I*	*II*	*III*	
w	6	6	6	4	4	4	34,000
x	5	5	5	2	2	2	18,000
y	8			3			10,000
z		7			3		3,500
z′			10			4	14,000
Unit profit in Rupees.	1·20	1·05	0·95	0·65	0·45	0·50	

14. The only survivor of a shipwreck was the doctor, who found himself on a desert island with some crates of prunes and a large quantity af cheese. The island has plenty of fresh water and coconuts, but no other food. The doctor knew he needed a minimum of 72 gms. of protein, 95 gms. of fat and 296 gms. of carbohydrate a day.

He thought the food values in grams of his supplies (per 30 gms.) were :—

	Protein	*Fat*	*Carbohydrates*
Cheese	7	12	0
Prunes	1	0	14
Coconut	2·5	9	4

(*i*) What daily diet would give his exact minimum requirements ?

(*ii*) His cheese and prunes would clearly last longer if he ate more nuts. If he was prepared to eat 330 gms. of coconut a day, and wished to eat as few prunes as possible, what should his diet be ? How much fat would he then have daily ?

15. A rectangular board has to be constructed so that its perimeter is less than 40 metres and greater than 24 metres. The ratio of the adjacent sides must be greater than 1 : 1 and less than 2 : 1. What integral requirements satisfy these requirements ?

16. A dictator seizes power in a small state and proceeds to plan the economy and labour forces. He discovers that there are two motor corporations. Each factory owned by corporation A makes each week 30 vans, 10 saloon cars and 10 lorries, while each factory in corporation B makes 10 vans, 10 saloon cars and 40 lorries weekly. He finds that the average combined home and overseas market for these vehicles is at least 100 vans, 60 saloon cars and 120 lorries weekly, but that previously these demands have been greatly exceeded in some respects and not met in others. In this organisation, how many factories in each corporation should continue to operate ?

If the labour force in each factory of A is half that in each factory of B, and if profits are directly proportional to labour force, which answer is best :

(*a*) for economising on labour ?
(*b*) for maximising profits ?

17. A firm is considering two medias for advertising their goods, say radio and cinema. The data known are :

	Radio	*Cinema*	*Required Levels*
Size of the audience	A_1	A_2	A
Average number noting the advertisement	B_1	B_2	
Average intelligence of the audience	I_1	I_2	I
Average income of audience	S_1	S_2	S
Proportion of females in the audience	F_1	F_2	F
Average age of the audience	G_1	G_2	G
Cost per insertion (in rupees)	C_1	C_2	

Set up a model to determine how often the two media should be used in order to minimise the advertising costs.

Solve the model when :

	B_j	A_j	I_j	S_j	F_j	G_j	C_j
Radio	0·3	$3\cdot5\times10^{6}$	3	4	0·8	40	8,000
Cinema	0·4	5×10^{6}	7	8	0·1	20	12,000

and $A=10^{7}$, $I=4$, $S=5$, $F=0\cdot4$, $G=30$.

18. A colliery manager works a number of faces and may do so using either of two machines, *A* and *B*.

	Output/day	*No. of face*	*Ash per cent*	*Production cost/ton*
Machine *A*	400 tons	20	10	Rs. 450·00
Machine *B*	200 tons	15	7	Rs. 500·00

He has to meet the following conditions :

Total output ranging between 2500 to 300 tons,

Total number of face men less than 180,

Ash content of product less than 9 percent.

The coal meeting this target would yield him Rs. 600·00 per ton. How many faces should he work by each method to maximise the overall profit.

19. Part of a farm is to be planted with wheat and kale. Observing the following restrictions

	Wheat	*Kale*	*Maximum Total*
No. of days per acre	2	1	10
Cost for labour per acre	7	6	12
Cost of fertilisers per acre	3	4	24

(*i*) find the greatest acreage that can be planted.

(*ii*) assuming a profit of Rs. 240 00 per acre for wheat and Rs. 180·00 per acre for kale, what acreage should be planted ?

20. A textile mill has L looms and S spindles. The spindles are capable of spinning two types of yarn, coarse and fine and the looms can weave two types of clothes, coarse and fine. Some of the yarn spun by the spindles is used in the mill and the rest is sold out. In case of shortage of yarn for the looms, it can be brought from other spinning mills. The following data is available.

b_1=number of spindles required to produce coarse yarn for one loom.

b_2=numbers of spindles required to produce fine yarn for one loom.

p_1=profit when one loom and its associated spindles prepare coarse cloth for one day.

p_2=profit when one loom and its associated spindles prepare fine cloth for one day.

q_1=profit when one spindle spins coarse yarn for one day.

q_2=profit when one spindle spins fine yarn for one day.

The problem is to allocate spindles and looms to the two varieties so as to maximise the overall profit.

Solve this problem in the specific case when :

$b_1=100$, $b_2=120$, $p_1=$Rs. 10, $p_2=$Rs. 12, $q_1=$Rs. 5, $q_2=$Rs. 3.

21. A greeting card dealer has three kinds of packets. Packet A contains 3 get-well cards, 1 birthday card, and 7 anniversary cards, Packet B contains 2, 3 and 1 respectively, Packet C contains 5, 4, 2 respectively. He wishes to merge these so as to form new packets, each containing one get-well card, one birthday card, and one anniversary card.

What is the least number of packets A, B, and C that must be used to accomplish this ? How many new packets will be made ?

[**Hint.** Suppose that x, y and z packets of A, B, C respectively are used to make t packets of new mixture. Then we must have

$$\begin{aligned} 3x+2y+5z&=t \\ x+3y+4z&=t, \\ 7x+y+2z&=t. \end{aligned}$$

On solving, we get $x=\frac{2t}{21}$, $y=\frac{5t}{21}$, $z=\frac{t}{21}$

We now observe that $t=21$ is the smallest value for which x, y and z are integers. Thus 21 new packets are made by using 2, 5 and 1 packets of A, B and C respectively.]

22. A company has 16 tons of a commodity of which 10 tons are in one warehouse W_1 and 6 tons in another warehouse W_2. It must all be delivered to three company stores S_1, S_2, S_3 in the amount 5 tons, 7 tons, 4 tons, respectively. The cost (in rupees) per ton of transporting from the two warehouses to the three stores is given by the table:

	S_1	S_2	S_3
W_1	7	5	8
W_2	2	3	4

Find the amounts to be shipped from each warehouse to each store in order to make the transportation cost minimum, and find the minimum cost.

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